

Quantum Thermodynamics with Missing Reference Frames: Decompositions of Free Energy Into Non-Increasing Components

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If an absolute reference frame with respect to time, position, or orientation is missing one can only implement quantum operations which are covariant with respect to the corresponding unitary symmetry group G . Extending observations of Vaccaro *et al.*, I argue that the free energy of a quantum system with G -invariant Hamiltonian then splits up into the Holevo information of the orbit of the state under the action of G and the free energy of its orbit average. These two kinds of free energy cannot be converted into each other. The first component is subadditive and the second superadditive; in the limit of infinitely many copies only the usual free energy matters.

Refined splittings of free energy into more than two independent (non-increasing) terms can be defined by averaging over probability measures on G that differ from the Haar measure.

Even in the presence of a reference frame, these results provide lower bounds on the amount of free energy that is lost after applying a passive covariant channel. If the channel properly decreases one of these quantities, it decreases the free energy necessarily at least by the same amount, since it is unable to convert the different forms of free energies into each other. For instance, if an electrical, optical, or acoustical signal loses some time accuracy after it has passed a passive time-invariant device, the results provide lower bounds on the free energy lost in the latter.

1. INTRODUCTION

Free energy is among the most important concepts of thermodynamics. In order to explain which aspects of free energy are relevant for this paper we recall that it

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is defined as⁽¹⁾

$$F := \langle E \rangle - k_B T S, \quad (1)$$

where $\langle E \rangle$ is the average of the inner energy of the system, k_B is the Boltzmann constant, T the temperature of an environment (where the definition refers to), and S is the entropy. Since we discuss quantum thermodynamics we shall set

$$\langle E \rangle := \text{tr}(\rho H),$$

where ρ is the density matrix of the considered quantum system and H its Hamiltonian (both acting on a Hilbert space \mathcal{H}). We shall replace S with the von-Neumann entropy²

$$S(\rho) := -\text{tr}(\rho \ln \rho),$$

and obtain the free energy

$$F(\rho) := \text{tr}(\rho H) - k_B T \text{tr}(\rho \ln \rho). \quad (2)$$

In thermal equilibrium one obtains⁽²⁾

$$F(\gamma_T) = -k_B T \ln \text{tr} \left[\exp \left(-\frac{H}{k_B T} \right) \right].$$

The relevance of the free energy is, among others, due to the following properties.

1. It is a functional whose minimum defines the thermal equilibrium state, i.e., the Gibbs state

$$\gamma_T := \frac{e^{-H/(k_B T)}}{\text{tr} \left(e^{-H/(k_B T)} \right)}. \quad (3)$$

The free energy difference between a non-equilibrium state and this minimum defines therefore a measure for the deviation from equilibrium. The typical return of a nearly equilibrium system to equilibrium (induced by interactions with a thermal bath) is hence accompanied by a monotonous decrease of free energy.

2. The difference of the free energy of the system to its equilibrium value describes the worth of the system as an energy source.
3. Free energy is a generating function in the following sense. Consider the perturbed equilibrium state obtained by replacing the Hamiltonian H

²Note that it is not straightforward to replace the classical quantity S with von-Neumann entropy in quantum mechanics.⁽³⁾ However, it goes beyond the scope of this article to discuss this issue. We assume here that the free energy in Eq. (2) is nevertheless a reasonable quantity for quantum thermodynamics.

with $H + \epsilon A$ for some self-adjoint operator A . Then the derivative of the function $\epsilon \mapsto F_\epsilon$ with

$$F_\epsilon := -k_B T \ln \text{tr} \left[\exp \left(- \frac{H + \epsilon A}{k_B T} \right) \right],$$

coincides with the expectation value of the observable A in the considered equilibrium state (see e.g. Ref. 2 for a derivation within a very general setting).

The issues discussed in this article refer mainly to item (1) and, even more importantly to item (2). The latter formalizes the fact that the usability of energy resources depends not only on the amount of energy contained in the system but also on the *entropy*. Roughly speaking, the idea is that in a typical energy conversion process the target system should be provided with energy without transferring entropy (like increasing the kinetic or potential energy of a macroscopic body, for instance). Therefore, the worth of a system for being an energy source depends on whether one can extract energy from it without releasing too much entropy S . This is because the transfer of the dispensable entropy to the environment requires the additional amount $S k_B T$ of energy when T is the temperature of the heat reservoir where the entropy is transferred to, e.g., the environment. Hence the amount of work that can be extracted from a physical system is not given by its inner energy. Instead, it depends also on the entropy and on a fixed reference temperature, namely the temperature of the environment which is potentially used as an entropy sink. Conversely, a system that has no inner energy at all (like a degenerate two-level system) can be used to extract energy from the environment if its physical state is not the maximal entropy state. In other words, *information* can directly be used to extract work from the surrounding heat bath.^(4,5,6) This fact implies on the other hand that the initialization of bits requires energy resources, an observation which is usually referred to as Landauer’s principle (cp.^(7,8)). All these statements can be brought into a consistent picture by the notion of free energy: for instance, the initialization of a *degenerate* two-level system to a well defined pure state increases its free energy and requires therefore resources of free energy.

To make the connection to information theory sharper we shall in the following redefine free energy by subtracting the equilibrium value, i.e.,

$$F(\rho) := \text{tr}(\rho H) - S(\rho) k_B T - \text{tr}(\gamma_T H) + S(\gamma_T) k_B T, \tag{4}$$

implying that $F(\rho) = 0$ if and only if $\rho = \gamma_T$. It is known that $F(\rho)$ is then (up to the constant $k_B T$) the Kullback-Leibler distance between ρ and γ_T ⁽⁹⁾

$$K(\rho || \gamma_T) := \text{tr}(\rho \ln \rho) - \text{tr}(\rho \ln \gamma_T).$$

In the sequel we will focus on so-called passive physical processes and define them as all those that use no additional energy resources for the implementation.

This assumption formalizes only the fact that we want to include all energy sources explicitly into the description. All passive completely positive trace-preserving operations G leave the Gibbs state invariant, i.e., $G(\gamma_T) = \gamma_T$. We conclude that free energy is monotonous under all such processes since no operation can increase the Kullback-Leibler distance between two density operators.⁽⁹⁾

Even though the monotonicity of free energy is maybe the most important constraint on the possible operations one should not forget that additional constraints arise in particular for quantum systems. However, they depend on additional assumptions on the set of physical processes. Refs. 5, 6, 10, 11 consider work extraction from quantum systems by unitary operations. In these models, the amount of extractable work does not only depend on the inner energy and the entropy of the system. Instead, it can only be calculated from more detailed information on the spectrum of the density operator.

In Ref. 12 we have furthermore considered *timing information* as a kind of thermodynamic resource. The idea was the following. Defining a physical system that includes all available clocks, no operation on the joint system can increase the information about an externally defined time reference frame. Every quantum state that is prepared in a superposition of different energy eigenstates with well-defined phase with respect to the external reference frame, provides some information on the latter. The impossibility to create information on the external time can also be interpreted as the impossibility to prepare superpositions of different energy eigenstates with well-defined phase. Such kind of coherent superpositions are therefore a special kind of deviation from equilibrium which could be considered as a resource in its own right. Whereas the complexity for communicating reference frames for time, position, and orientation has been extensively studied in the literature (e.g. Refs. 13 and 14) and its cryptographic power has been pointed out,⁽¹⁵⁾ the thermodynamical relevance of reference frames is not yet well understood. Ref. 16 considers the thermodynamic cost of establishing reference frames with classical communication and the authors in Ref. 17 observe that the worth of thermodynamic resources is reduced by missing reference frames. The latter observation is the venue of this article.

The paper is organized as follows. In Sec. 2 we sketch the idea to restrict the set of operations to covariant maps, i.e., those that can be implemented without referring to an external frame. We rephrase the idea of Vaccaro *et al.*⁽¹⁷⁾ to consider a thermodynamic theory that is modified by the additional constraint of covariance.

In Sec. 3 we will show explicitly that the time covariance condition implies a splitting of free energy into two terms which cannot be converted into each other. We refine this splitting of free energy into arbitrarily many terms reflecting the fact that different kind of timing information that refer to different *time scales* cannot be converted into each other. The theory can be generalized to other covariance conditions that may stem, for instance, from missing spatial or rotational reference frames provided that the considered unitary symmetry operation commutes with

the Hamiltonian. In Sec. 4 we argue why the splitting loses its relevance in the macroscopic limit of a large number of identical systems. In Sec. 5 we will show that the results have implications also for situations where reference frames are available. This is because every covariant operation that decreases one kind of free energy decreases also the total amount of free energy if no additional source of free energy is available. This loss is clearly irreversible even if a reference frame is available. We sketch how to apply this idea to time-invariant passive devices (i.e. devices without energy source) in optics or in electrical engineering. Then our results imply that a device that causes an indeterministic time delay of the output signal causes necessarily a loss of free energy.

2. COVARIANT OPERATIONS

The set of possible operations on a quantum system is given by the set of completely positive trace-preserving maps.⁽¹⁸⁾ In Ref. 12 we have argued that not every CP map C can be implemented if no time reference frame is available. Consider a quantum system with free evolution

$$\alpha_t(\rho) = e^{-iHt} \rho e^{iHt}.$$

Assume that the state ρ has been prepared at time $t = 0$. If a person implements an operation C at time instant s the state of the system after C has been implemented is described by

$$C(\alpha_s(\rho)).$$

Looking at the system later it is described by the state

$$\alpha_{t-s}(C(\alpha_s(\rho))), \tag{5}$$

if t is the time that has passed since the system was prepared. If no clock was available during the implementation of C , it is implemented at a random time instant s . We may therefore define a CP map \overline{C} that results from averaging (5) over all $0 \leq s \leq t$. If the quantum system has discrete spectrum and t is large compared to the time scale given by the inverse of the minimal distance between its energy eigenvalues, \overline{C} satisfies approximately the covariance condition

$$\alpha_t \circ \overline{C} = \overline{C} \circ \alpha_t, \quad \forall t. \tag{6}$$

In Ref. 12 we have thus assumed that the set of operations which can be performed without additional clock is given by those that satisfy the above covariance condition (6). In Ref. 19 we have analyzed this class of CP maps in full detail.

With the same arguments one can restrict the set of available operations to those satisfying covariance conditions with respect to other symmetry groups if

the corresponding reference frame is not available.³ Interesting instances are given by the group of space translations or by the rotation symmetry.⁽¹⁷⁾ The symmetry is represented by a unitary group $U_g, g \in G$ acting on the Hilbert space of the considered system. We assume that $[U_g, H] = 0$, otherwise (U_g) would not be a symmetry group of the Hamiltonian and the missing reference frame would even make the definition of H impossible. Every transformation is covariant with respect to G , i.e., we can only implement a CP map C with

$$C(U_g \rho U_g^\dagger) = U_g C(\rho) U_g^\dagger. \quad (7)$$

It has already been observed in Ref. 17 that the absence of a reference frame puts thermodynamically relevant constraints on the set of available operations. The idea is that the system may contain some information that is not accessible without using the frame. The authors assume that the work extractable from a d -dimensional system being in the mixed state ρ is usually (if a reference frame is available) given by

$$k_B T (\ln d - S(\rho)). \quad (8)$$

Note that this definition of extractable work refers actually to thermodynamics in degenerate systems or the infinite temperature limit, where the Hamiltonian of the system is irrelevant and the free energy $F(\rho)$ is given by the difference of the entropy to the maximally mixed state. Within this thermodynamic perspective⁽²¹⁾ all maximally mixed states are free resources whereas usual (finite temperature and non-degenerate) thermodynamics assumes all Gibbs states to be free and “worthless” resources. Note that Eq. (8) is, up to the constants $k_B T$, the Kullback-Leibler distance $K(\rho||\omega)$ if $\omega := \mathbf{1}/d$ is the maximally mixed state. We may therefore consider the usual expression of free energy as the finite temperature analogue of (8). Hence it is slightly more general to consider F as defined in Eq. (4) as the extractable work. We will refer to these two points of view as the finite and the infinite temperature picture, respectively.

If no reference frame is available, the extractable work in the infinite temperature picture reduces to

$$W_G := k_B T (\ln d - S(\bar{\rho})) \quad (9)$$

instead of Ref. 17 (as the authors of Ref. 17 observe), where

$$\bar{\rho} := \int_G U_g \rho U_g^\dagger d\mu(g)$$

is the average of ρ over the orbit of G . Here we have implicitly assumed for the moment that G is a compact group and denoted its Haar measure by μ .

³ For a model for a formulation of quantum mechanics that avoids absolute reference frames see Ref. 20.

Introducing the reference information by

$$R(\rho) := S(\bar{\rho}) - S(\rho), \tag{10}$$

the deficit between the terms (8) and (9) is $R(\rho)k_B T$, a term called asymmetry in Ref. 17. The asymmetry is non-increasing under covariant operations since no trace-preserving CP map can increase the Holevo information.⁴ The authors of Ref. 17 prove that even covariant operations that include measurements cannot increase the average asymmetry as long as the probabilities for the measurement outcomes are G -invariant. They consider a family C_j of maps where each C_j is a covariant CP map⁵ and show that the average asymmetry of the conditional post measurement states $\rho_j := C_j(\rho)/p_j$ with $p_j := \text{tr}(C_j(\rho))$ cannot exceed the initial asymmetry. However, this generalization needs not explicitly be made when we include a toy version of a measurement apparatus into the description as follows. Let $|0\rangle\langle 0|$ denote the initialized pointer of some apparatus. One can check that there exists a covariant map C that transfers the state

$$|0\rangle\langle 0| \otimes \rho \tag{11}$$

of the “measurement apparatus” plus system into

$$\sum_j |j\rangle\langle j| \otimes C_j(\rho), \tag{12}$$

where the state $|j\rangle$ indicates that j was measured. Assuming that the group acts trivially on the ancilla system, the asymmetry of the state (12) coincides with the average asymmetry of the ensemble ρ_j, p_j . Since we know that C cannot increase the asymmetry of (11), the average post-measurement asymmetry cannot be increased either.

In the following section we will show that the observations of Ref. 17. can be generalized to the finite temperature setting and give rise to two kinds of free energy.

3. DECOMPOSITION OF FREE ENERGY

The key statement of this section is that the different components in which we decompose the free energy are independent resources in the sense that no covariant channel can increase them without access to an additional energy resource. To state this formally, we will use the notion of a passive channel:

⁴This follows, for instance, if one rewrites Holevo information as the mutual information of the bipartite state $\sum_j p_j |j\rangle\langle j| \otimes \rho_j$. Then the statement follows because no local operation on one system can increase the mutual information⁽²²⁾ of the joint system.

⁵Note that this class of operations does *not* include general covariant measurements where the outcome probabilities change according to the group action.

Definition 1. A trace-preserving CP map C acting on a quantum system with Hamiltonian H is called *passive* if $C(\gamma_T) = \gamma_T$ with the thermal state γ_T as defined in Eq. (3).

We have already seen that passivity implies $F(C(\rho)) \leq F(\rho)$. To define our decomposition of free energy, we assume for the moment that the considered quantum system has discrete energy spectrum such that the time average $\bar{\rho}$ exists. It is then given by

$$\bar{\rho} := \sum_j P_j \rho P_j,$$

where (P_j) is the family of energy eigenprojections. We write the free energy $F(\rho)$ as

$$F(\rho) = F(\rho) - F(\bar{\rho}) + F(\bar{\rho}),$$

and use the fact that averaging over the time can only decrease the free energy since the energy term in (4) remains the same. Then we have

$$F(\rho) - F(\bar{\rho}) = (S(\bar{\rho}) - S(\rho)) k_B T = R(\rho) k_B T,$$

with $R(\rho)$ as in Eq. (8). We conclude

$$F(\rho) = R(\rho) k_B T + F(\bar{\rho}),$$

where we call $F(\bar{\rho})$ the *covariant free energy*. Note that $F(\bar{\rho})$ can be considered as the natural generalization of the accessible work W_G in Eq. (9) to our finite temperature setting. To see that $F(\bar{\rho})$ is non-increasing when applying passive channels we observe that a covariant channel C that converts a state ρ to another state σ must necessarily convert $\bar{\rho}$ to $\bar{\sigma}$. The channel C is therefore only passive if $F(\bar{\rho}) \geq F(\bar{\sigma})$. This shows that asymmetry as well as covariant free energy are both non-increasing under passive covariant operations. We rephrase these observations as a theorem:

Theorem 1. The free energy of a quantum system with discrete energy levels can be decomposed into

$$F(\rho) = R(\rho) k_B T + F(\bar{\rho}),$$

where

$$R(\rho) := S(\bar{\rho}) - S(\rho)$$

is the Holevo information of the time orbit and $F(\bar{\rho})$ is the free energy of the orbit average. The terms $R(\rho)$ and $F(\bar{\rho})$ are both non-negative and non-increasing with respect to time-covariant passive operations.

Theorem 1 can be generalized in two respects. First, we may have an arbitrary group representation instead of the time evolution provided that it leaves the Hamiltonian invariant. Then the term $tr(\rho H)$ is preserved by averaging, too. Second, we need not necessarily consider uniform averaging over the whole group. Instead, we can define hierarchies of states, obtained by averaging more and more over the group, and calculate free energy differences between more and less mixed states. By this procedure, we obtain a splitting of free energy into many independent terms. We phrase this idea also as a theorem:

Theorem 2. *Given a quantum system with Hilbert space \mathcal{H} and Hamiltonian H . Let $g \mapsto U_g$ with $g \in G$ be the unitary representation of a group G acting on \mathcal{H} such that $[U_g, H] = 0$. Let $\mu_1, \mu_2, \dots, \mu_n$ be an n -tuple of probability measures on G such that there exist measures ν_j on G with $\mu_j * \nu_j = \mu_{j+1}$, i.e., μ_{j+1} is the convolution⁶ of μ_j with a third measure ν_j . Let μ_1 be the Dirac measure on the identity element of G .*

Let A_μ be the CP map given by the average

$$A_\mu(\rho) := \int_G U_g \rho U_g^\dagger d\mu(g).$$

Then the free energy $F(\rho)$ splits up into the n terms

$$F(\rho) = \sum_{j=1}^n F_j(\rho),$$

with

$$F_j(\rho) := F(A_{\mu_j}(\rho)) - F(A_{\mu_{j+1}}(\rho)) \quad j = 1, \dots, n - 1.$$

and

$$F_n(\rho) := F(A_{\mu_n}(\rho)).$$

All terms $F_j(\rho)$ for $j = 1, \dots, n$ are non-negative and non-increasing with respect to passive covariant operations.

Proof: $F_n(\rho)$ is clearly non-negative. To see that it is non-increasing we observe

$$F(A_{\mu_n}(C(\rho))) = F(C(A_{\mu_n}(\rho))) \leq F(A_{\mu_n}(\rho)),$$

where the last inequality is due to the monotonicity of usual free energy under passive operations. The terms $F_j(\rho)$ for $1 \leq j \leq n - 1$ are, up to the constant

⁶The convolution product $\mu * \nu$ of measures μ, ν on G is here defined by the probability distribution of $h \circ g$ if $g \in G$ and $h \in G$ are independently distributed according to μ and ν , respectively (see Ref. 3, and adapt Def. 19.8 to our setting).

$k_B T$, given by the entropy difference

$$S(A_{\nu_j}(\rho_j)) - S(\rho_j), \quad (13)$$

with

$$\rho_j := A_{\mu_j}(\rho).$$

This follows easily from $A_{\mu * \nu} = A_\nu \circ A_\mu$. Expression (13) is for fixed j the Holevo information⁽¹⁸⁾ of the ensemble defined by the family of states $U_g \rho_j U_g^\dagger$ with $g \in G$ according to the probability measure ν_j . It is therefore non-negative. To show that it is non-increasing when applying C we observe that the covariance implies

$$A_{\mu_j}(C(\rho)) = C(\rho_j),$$

and $F_j(C(\rho))$ is therefore, up to the constant $k_B T$, the Holevo information of the ensemble

$$U_g C(\rho_j) U_g^\dagger = C(U_g \rho_j U_g^\dagger), \quad g \in G$$

according to the probability measure ν_j . Then monotonicity of $F_j(\rho)$ with respect to C follows again from the monotonicity of Holevo information. \square

The advantage of Theorem 2 compared to the preceding remarks is not only that it allows a splitting into more than two terms. It is furthermore important that it allows a splitting for non-compact groups since it does not refer to a uniform average over the whole group.

Note that there are also other ways of splitting the free energy. Consider a passive CP map L that is a convex sum of unitary maps, i.e.,

$$L(\rho) := \sum_j p_j U_j \rho U_j^\dagger,$$

where the values p_j define a probability measure on some set $\{U_j\}$ of unitaries. Assume that L satisfies the covariance condition (6). Then we may clearly split the free energy into

$$F(\rho) = F(L(\rho)) + kT(S(L(\rho)) - S(\rho)),$$

and both terms are non-increasing under a map C that commutes with the ‘‘coarse graining operation’’ L . The operation C cannot increase the first term because $F(L(\rho))$ is a Kullback-Leibler distance to $\gamma_T = L(\gamma_T)$. It cannot increase the second term because this expression is a Holevo information.

The following example shows that the coarse graining may, for instance, refer to a kind of *macroscopic* description level. Consider n distinguishable two level

systems being in a joint state ρ . From a macroscopic point of view, one may not be able to distinguish between them and consider the state

$$L(\rho) := \frac{1}{n!} \sum_{\pi \in S_n} \pi(\rho),$$

where S_n denotes the permutation group on n elements and π the unitary representation given by the permutation of the qubits. Then every map that acts on a permutation symmetric way cannot increase either of both quantities. In other words, when restricted to “macroscopic actions,” both parts are independent resources. Since the goal of this article is to find the restrictions which are imposed by the covariance condition alone we shall not discuss this generalization any further.

4. SUPERADDITIVE AND SUBADDITIVE COMPONENTS

We will now restrict our attention again to the splitting into two free energy terms like in the beginning of Sec. 3 and investigate how these quantities behave when systems are composed to joint systems. Let us consider a two-level system with lower and upper state, denoted by $|0\rangle$ and $|1\rangle$, respectively. Let E denote the energy gap between both levels. If $k_B T \gg E$ the equilibrium state is almost the maximally mixed state $\gamma_\infty := \mathbf{1}_2/2$. The free energy of the state

$$|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

is given by

$$F(|+\rangle\langle+|) = k_B T (S(\overline{|+\rangle\langle+|}) - S(|+\rangle\langle+|)) + F(\gamma_\infty) = \ln 2 k_B T + F(\gamma_\infty).$$

One checks easily that $F(\gamma_\infty)$ is negligible compared to the first term since $E \ll k_B T$ and the entropy difference between γ_∞ and γ_T is small.

To see the asymptotics of many copies of $|+\rangle\langle+|$ we observe that the entropy of the time average of $|+\rangle\langle+|^{\otimes n}$ is exactly the entropy of the binomial distribution $B_{n,1/2}$ with

$$B_{n,1/2}(k) := \frac{1}{2^n} \binom{n}{k}.$$

Hence the entropy of the average of $|+\rangle\langle+|^{\otimes n}$ increases only with $O(\ln n)$ since the measure is supported by only n different points. In other words, there are only n different eigenspaces of the joint Hamiltonian

$$H = E \sum_j \sigma_z^{(j)},$$

where $\sigma_z^{(j)}$ is the Pauli matrix σ_z acting on qubit j . For the same reasons, the covariant free energy of $|+\rangle\langle+|^{\otimes n}$ can be bounded from below by $n \ln 2 - O(\ln n)$. Hence the asymmetry part of the free energy is for large n more and more a negligible fraction of the total free energy. Similar arguments apply to the general situation.

This does not mean, however, that the splitting is completely irrelevant for large particle numbers. Instead, the results show that every passive process that increases the covariant free energy of some of the particles requires interactions between them since separate processes on some part of the particles would not be able to increase their covariant free energy. The process is only possible if neither the covariant free energy nor the reference information of each subset of interacting particles is increasing. Therefore our results imply statements on the *complexity* of the considered process. This kind of complexity issues are related to the questions discussed in Ref. 24 where we have discussed the complexity of molecular heat engines. The key observation was that the additional constraints for energy conversion processes that arise in simple quantum systems imply statements on the complexity of energy conversion processes in macroscopic ensembles of particles.

For the sake of completeness we sketch the proof of the superadditivity of covariant free energy. Then the subadditivity of the asymmetry term follows because total free energy is additive. The essential observation is that for two states ρ and σ with time evolution α_t and β_t , respectively, the entropy of the joint time average $\overline{\rho \otimes \sigma}$ cannot be greater than the entropy of the tensor product of the averages, i.e., $\overline{\rho} \otimes \overline{\sigma}$. This is because the latter state can be obtained from the former by averaging over all possible *relative* time translations $\alpha_t \otimes \beta_{-t}$. We have therefore

$$F(\overline{\rho \otimes \sigma}) \geq F(\overline{\rho} \otimes \overline{\sigma}) = F(\overline{\rho}) + F(\overline{\sigma}).$$

5. APPLICATIONS

Remarkably, the results above have also implications for situations where a reference frame is available because every passive covariant operation that decreases one term $F_j(\rho)$ (see Theorem 2) necessarily decreases the total free energy since it is unable to convert one kind of free energy into the other. This can be used to derive lower bounds on the loss of free energy of a physical signal like an electrical pulse or a light pulse when it passes a device such that it degrades the time accuracy of the pulse (see Fig. 1). We will explain this idea using a system with discrete energy levels as a toy model for the physical signal.

Consider for instance a system which has some integer values n_1, n_2, \dots, n_d as energy spectrum. Due to the periodicity, we can restrict its group of time translations to $G := SU(1)$. Parameterizing G by the interval $[0, 2\pi]$ we obtain

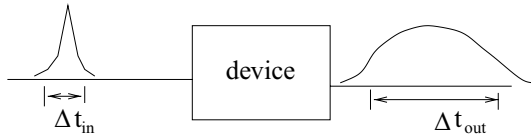


Fig. 1. Symbolic drawing of input and output signals of a device that corrupts the localization in time. The curves do not necessarily have direct physical meaning. Their widths only symbolize the time scale of distinguishability and how it is changed. The curves could, however, have a direct meaning in the following situation. Consider a pulse which has on the considered time scale a well-defined time of arrival (when assuming that the quantum uncertainty⁽²⁵⁾ may be only relevant on a much smaller scale) which is unknown because it fluctuates according to a classical probability distribution. Then the curves may be interpreted as such a probability measure on the time axis. Assume that the device degrades the time accuracy in the sense that the distribution is strongly smeared out ($\Delta t_{out} \gg \Delta t_{in}$). According to our results, this leads necessarily to a loss of free energy.

the unitary representation

$$t \mapsto U_t := \text{diag}(e^{-itn_1}, e^{-itn_2}, \dots, e^{-itn_d}).$$

Now we consider an input state ρ which is perfectly distinguishable from its time evolved state $\alpha_s(\rho)$ for some $s \in \mathbb{R}$ in the sense that

$$\text{tr}(\rho \alpha_s(\rho)) = 0.$$

Let C be a passive time covariant operation that corrupts the timing information of ρ in the sense that the corresponding output density matrices $C(\rho)$ and $C(\alpha_s(\rho)) = \alpha_s(C(\rho))$ are not perfectly distinguishable any longer. Then we can use the results of Sec. 3 to bound the free energy loss of the channel from below as follows. Define a measure on $SU(1)$ by

$$\mu := \frac{1}{2}(\delta_0 + \delta_s),$$

where δ_0 and δ_s denote the Dirac measures at the time instants $t = 0$ and $t = s$, respectively. By applying Theorem 2 using the measures $\mu_1 := \delta_0$ and $\mu_2 := \mu$, the free energy of the input splits up into

$$F(\rho) = k_B T (S(A_\mu(\rho)) - S(\rho)) + F(A_\mu(\rho)) = k_B T \ln 2 + F(A_\mu(\rho)),$$

and for the output into

$$F(C(\rho)) = k_B T (S(A_\mu(C(\rho))) - S(C(\rho))) + F(A_\mu(C(\rho))).$$

The fact that the output states are not perfectly distinguishable is equivalent to the statement that the Holevo information of an ensemble that consists of the states $C(\rho)$ and $C(\alpha_s(\rho))$ with probability $1/2$ each is strictly less than $\ln 2$, i.e., $c := S(A_\mu(C(\rho))) - S(C(\rho)) < \ln 2$. Then the loss of free energy satisfies

$$F(\rho) - F(C(\rho)) \geq (\ln 2 - c) k_B T ,$$

since

$$F(A_\mu(C(\rho))) = F(C(A_\mu(\rho))) \leq F(A_\mu(\rho)).$$

The intuitive content of this statement is that it provides lower bounds on the free energy loss caused by devices that corrupt the time accuracy of the input signal by generating output signals with stochastic time delay.

To consider a more concrete physical situation, assume some electrical, acoustical, or optical signal enters a passive device whose input-output behavior is described by the time covariant map C . The time covariance formalizes only the fact that the state of the physical device is stationary before the signal enters into it (see Ref. 19 for details). Assume that the channel converts some input signal that can be perfectly distinguished from its time evolved copy that is defined by a time shift Δt in the sense that the quantum states ρ and $\alpha_{\Delta t}(\rho)$ are mutually orthogonal density operators. Assume that the channel generates an unknown time delay such that the output density operators $C(\rho)$ and $C(\alpha_{\Delta t}(\rho))$ are not perfectly distinguishable.

Then the fluctuation of the time delay leads necessarily to a loss of free energy. One may argue that this conclusion would also be obvious without using our result since the fluctuations increase the entropy of the state. However, a priori it is not clear whether the channel could change the inner energy of the signal such that the loss of free energy caused by an increase of entropy is compensated by an increase of the inner energy $\text{tr}(\rho H)$. The statement that no covariant channel can do such a compensation is the key statement of this article.

The signal above is assumed to pass the device only once. By applying our results to such a situation with aperiodic dynamics we have actually ignored the fact that it then refers necessarily to *continuous* spectrum. Otherwise the free evolution of the signal would be quasiperiodic. We may remove this inconsistency by choosing the spectrum such that the periods is large compared to the considered time scale. The problem with the aperiodic limit is anyway that it refers to an infinite amount of free energy. This can be seen as follows. Given a density operator ρ such that for some $t \in \mathbb{R}$ all states $\alpha_{nt}(\rho)$ for $n \in \mathbb{Z}$ are perfectly distinguishable from ρ . Then one can choose an arbitrary probability measure on \mathbb{Z} by (p_j) with $\sum_j p_j = 1$ and observe

$$F(\rho) - F\left(\sum_j p_j \alpha_{t_j}(\rho)\right) = k_B T S(p),$$

where $S(p)$ denotes the Shannon entropy of p . By choosing measures p with diverging entropy and recalling that F is always non-negative, the statement follows. However, even though the free energy diverges in the aperiodic limit our statement on the free energy *loss* still makes sense since the absolute value of the free energy is irrelevant in this context.

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REFERENCES

1. H. Callen. *Thermodynamics*. J. Wiley and Sons, New York, 1960.
2. O. Bratteli and D. Robinson. *Operator algebras and quantum statistical mechanics*. Springer, vol. 2, New York, 1987.
3. A. Allahverdyan and T. Nieuwenhuizen. Resolution of the Gibbs paradox via quantum thermodynamics. <http://xxx.lanl.gov/abs/quant-ph/0507145>.
4. L. Szilard. Über die Entropieverminderung in einem thermodynamischen System bei Eingriffen intelligenter Wesen. *Z. Phys.* 840–856 (1929).
5. D. Janzing, P. Wocjan, R. Zeier, R. Geiss and Th. Beth. Thermodynamic cost of reliability and low temperatures: Tightening Landauer's principle and the Second Law. *Int. Jour. Theor. Phys.* 39(12):2217–2753 (2000).
6. D. Janzing. A quasi-order of resources as a new concept for a thermodynamic theory of quantum state preparation. In *Sciences of the interface*, Tübingen, 2000. Genista Verlag.
7. R. Landauer. Irreversibility and heat generation in the computing process. *IBM J. Res. Develop.* 5:183–191 (1961).
8. C. Bennett. The thermodynamics of computation—a review. *Int. J. Theor. Phys.* 21:905–940 (1982).
9. M. Ohya and D. Petz. *Quantum entropy and its use*. Springer Verlag, 1993.
10. A. Allahverdyan, R. Balian and T. Nieuwenhuizen. Quantum thermodynamics: thermodynamics at the nanoscale. *Journ. Mod. Opt.* 51:2703–2711 (2004).
11. A. Allahverdyan, R. Balian and T. Nieuwenhuizen. Maximal work extraction from quantum systems. *Europhys. Lett* 67:565 (2004).
12. D. Janzing and T. Beth. Quasi-order of clocks and their synchronism and quantum bounds for copying timing information. *IEEE Trans. Inform. Theor.* 49(1):230–240 (2003).
13. G. Chiribella, G. D'Ariano, P. Perinotti and M. Sacchi. Efficient use of quantum resources for the transmission of a reference frame. *Phys. Rev. Lett.* 93:180503 (2004).
14. T. Rudolph and L. Grover. On the communication complexity of establishing a shared reference frame. *Phys. Rev. Lett.* 217905 (2003).
15. S. Bartlett, T. Rudolph and W. Spekkens. Decoherence-full subsystems and the cryptographic power of a private reference frame. *Phys. Rev. A* 70:032307 (2004).
16. D. Janzing and T. Beth. Synchronizing quantum clocks with classical one-way communication: Bounds on the generated entropy. <http://xxx.lanl.gov/abs/quant-ph/0306023v1>.
17. J. Vaccaro, F. Anselmi, H. Wiseman, and K. Jacobs. Complementarity between extractable mechanical work, accessible entanglement, and ability to act as a reference frame, under arbitrary superselection rules. <http://xxx.lanl.gov/abs/quant-ph/0501121>.
18. M. Nielsen and I. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2000.
19. D. Janzing. Decomposition of time-covariant operations on quantum systems with continuous and/or discrete energy spectrum. *Journ. Math. Phys.* 122107 (2005).
20. D. Poulin. Toy model for a relational formulation of quantum theory. <http://xxx.lanl.gov/abs/quant-ph/0505081>.
21. M. Horodecki, P. Horodecki, and J. Oppenheim. Reversible transformations from pure to mixed states, and the unique measure of information. *Phys. Rev. A* 67:062104 (2003).
22. G. Lindblad. Completely positive maps and entropy inequalities. *Comm. Math. Phys.* 40:147–151 (1975).

23. E. Hewitt and K. Ross. *Abstract Harmonic Analysis*, vol. I. Springer, 1963.
24. D. Janzing. On the computational power of molecular heat engines. *J. Stat. Phys.*, (Online DOI: 10.1007/s10955-005-8015-9), 2006.
25. Y. Aharonov, J. Oppenheim, S. Popescu, B. Reznik and W. Unruh. Measurement of time-of-arrival in quantum mechanics. *Phys. Rev. A* **57**:4130 (1998).